Quantum Corrections to the Period of the Non Linear Pendulum via Path Integrals

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Abstract. We treat the problem of the quantum pendulum in order to calculate the swing period of its classical counterpart. For this we apply the semiclassical path integral method to construct the system's fractional Green's function. The energy eigenvalues arise as the poles of the later, and the frequency of oscillation comes from the difference of two successive eigenvalues. The linear and non-linear potential are separately discussed and the role of the derivatives with respect to the energy of the action phase-factors, is discussed in detail.

INTRODUCTION

The simple pendulum is a famous case study in classical mechanics that still attracts attention, since it involves many problems in different branches of physics, leading to many interesting applications. Mentioning a few we can distinguish its diachronic and historical use as a clock or a metronome [1], as a seismometer [2], as a ballistic galvanometer [3], as a definite instrument for determining gravitational constant G [4] or an instrument for proving the equivalence of gravitational and inertial mass [5], part of an experimental setup for measuring viscosity [6], and many others.

It was only a couple of years after the breakthrough of quantum mechanics, that a study of the quantum pendulum appeared for the first time in science [7], by Edward Condon. Condon would soon become one of the most influential physicists in the quantum area with major contributions to atomic physics. The substantial difference in the quantum case comes from the possibility of the particle to exist in a classically inaccessible region, (barrier), where the potential energy exceeds the total energy, via the tunneling effect. Thus, quantum pendulum at most concerns with what is known as hindered or restricted rotation, which is the inhibition of rotation of molecular structures about a bond due to the presence of a sufficiently large rotational barrier, [8].

In addition, the dynamics of the driven classical pendulum resembles those of the pure quantum mechanical Josephson effect [9] in superconductive structures.

In the present work we are interested to the calculation of the period of the non linear pendulum by taking in account the peculiar quantum mechanical effect of tunneling. As far as pendulum is treated inside the frame of classical mechanics governed by the Newton's laws of motion, period calculation is straightforward. Through the linearization of the force term, valid in the small angle regime, we can easily solve the differential equation of motion. The solution of the later contains the well known frequency of oscillation, $\sqrt{g/l}$, in terms of gravity acceleration and the length of the pendulum, which is amplitude independent. This is what we call isochronism. On the other hand, when we proceed without making the linear approximation, the period is produced by integrating time through the energy conservation principle, in terms of the elliptic integral of the first kind, and depends on the amplitude. Unfortunately the period is not now given in a closed form.

As far as quantum pendulum is concerned we must first proceed to the construction of Schrödinger's equation. For this, we start from the classical Lagrangian and construct the classical Hamiltonian by finding the generalized momentum of the system. Then we apply the conventional rule by replacing the canonical momentum with the corresponding quantum operator. The solution of this Schrödinger equation is a difficult task, due to the non linear potential term, not only for the eigenfunctions but for the energy eigenvalues as well, and complicated arithmetic methods are concerned. Thus, the result lacks of any analytical relations that would demonstrate the qualitative involvement of tunneling to the swing period. However if we assume a parabolic form of the potential, (which of course is equivalent to the linearization of the classical force term), we reduce the complexity to the one of the harmonic oscillator problem. The classical frequency of oscillation is then explicitly contained in the quantum energy spectrum, equal to the photon frequency for transition between two adjacent states, and is energy independent. The later is equivalent to the forementioned classical isochronism.

Instead of trying to solve Schrödinger equation we can attack the problem in a different way. For this we construct the system's Green's function, as the Fourier transformed of its propagator, via the aid of semiclassical path integration. According to semiclassical path integration, the Green's function for one dimensional physical systems is fulfilled through the mutual contribution of all possible changes in phase of the wave-function due propagation or reflection. Again, this is not trivial at all since it demands the summation of all possible changes of phase, (real or imaginary), for the infinite set of possible paths, without omitting any of them, performing the tedious calculation while always keeping the correct order of regions interchange. This results in a complicated but compact formula for the Green's function. Thus, the great advantage comes from the fact that we can now reveal the analytic form of the energy spectrum in a closed form as the approximate complex poles of the fractional Green's function, keeping their real part. This is done after expanding the denominator, (constituted by transcendental functions), in terms of analytic tunneling parameters. Again, we extract the classical period from the difference of two adjacent energy eigenvalues. The energy spectrum turns to be discrete although still infinite. The spacing between adjacent energy levels is of the order of magnitude of Planck's constant ($\sim 10^{-34}$) times the frequency of classical oscillations. This corresponds to a truly negligible quantity either for a macroscopic pendulum or a microscopic. For example, taking the length of the pendulum to be comparable in magnitude with that of a chemical bond, (meaning some Angstroms), we get an energy spacing of about 10^{-9} eV. This means that even in the classical case where all values of energy are permitted, no one can in practice achieve such a

high accuracy in measuring the energy. This makes the spacing rather unimportant. In addition the arithmetic value of the quantum corrected swing period turns to be only negligibly different from the classical. Since this is the case, we do not proceed to any quantitative calculation but rather bring out the qualitative involvement of tunneling.

The structure of the present paper is as follows. In the next section we analyze in short the dynamics of the classical pendulum stressing the lack of isochronism for increasing amplitudes, caused by the non linear nature of the potential involved in the Langrangian. In the next section we introduce the Schrödinger equation for the pendulum and demonstrate its solutions through the Mathieu functions while pointing the quantum equivalent to classical isochronism. Following this we develop the semiclassical path integral method for the construction of the system's Green's function, which is the Fourier transformed of its propagator. We then proceed to the calculation of the energy spectrum and subsequently of its classical period of motion through the difference of two successive eigenvalues. We develop a prototype formula in a compact form. Finally we mark and further analyze the role of the derivatives with respect to the energy of the action phase-factors, as these appear in the above mentioned formula. In the last section we briefly conclude.

THE CLASSICAL PENDULUM

The simple pendulum governed by the laws of classical mechanics concerns a plane motion with only one degree of freedom. It consists of a point mass m attached to one end of a weightless chord of length l, moving without any resistance acted upon by gravity and the tension in the chord, while the other end turns without friction around a fixed point. Although simple, its exact solution is not trivial at all, due to the non linear force term that appears in its differential equation

$$ml\ddot{\theta} + mg\sin\theta = 0 \tag{1}$$

The exact solution of the above equation demands the aid of elliptic functions. However we can approach pendulum's vibrational motion through the linearized equation

$$ml\ddot{\theta} + mg\theta = 0 \tag{2}$$

valid for small only amplitudes where $\sin\theta \cong \theta$. The solution of (2) which is the differential equation of a harmonic oscillator is straight forward and gives the well known result for the period of swinging

$$T_o = 2\pi\left(l/g\right)^{1/2} \tag{3}$$

On the other hand the accurate result for the period is met by integrating time in the energy conservation principle, as far as Figure 1 is concerned according to which α is the angular amplitude and the minimum of potential energy is taken as zero for θ=0. Doing so we find

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) = mgl(1-\cos\alpha) \Rightarrow$$

$$\frac{d\theta}{dt} = \sqrt{2\frac{g}{l}(\cos\theta - \cos\alpha)} \Rightarrow dt = \frac{d\theta}{\sqrt{2\frac{g}{l}(\cos\theta - \cos\alpha)}} \Rightarrow \qquad (4)$$

$$T(\alpha) = 4\sqrt{\frac{l}{2g}}\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = 2\sqrt{\frac{l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}}$$

Following Landau's prescription [10] we substitute $\sin\xi = \dfrac{\sin(\theta/2)}{\sin(\alpha/2)}$ to convert the above relation to

$$T(\alpha) = 4\sqrt{\frac{l}{g}}\,K(\sin(\alpha/2)) \quad \text{or} \quad T(E) = 4\sqrt{\frac{l}{g}}\,K\left(\sin\left[\cos^{-1}(1 - E/mgl)/2\right]\right) \qquad (5)$$

where $K(k) = \displaystyle\int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - k^2\sin^2\xi}}$ is the complete elliptic integral of the first kind [11]. The two

relations are connected through the $T(\alpha) = T_o\dfrac{K(\sin(\alpha/2))}{\pi/2}$ and of course coincide for $\alpha \to 0$.
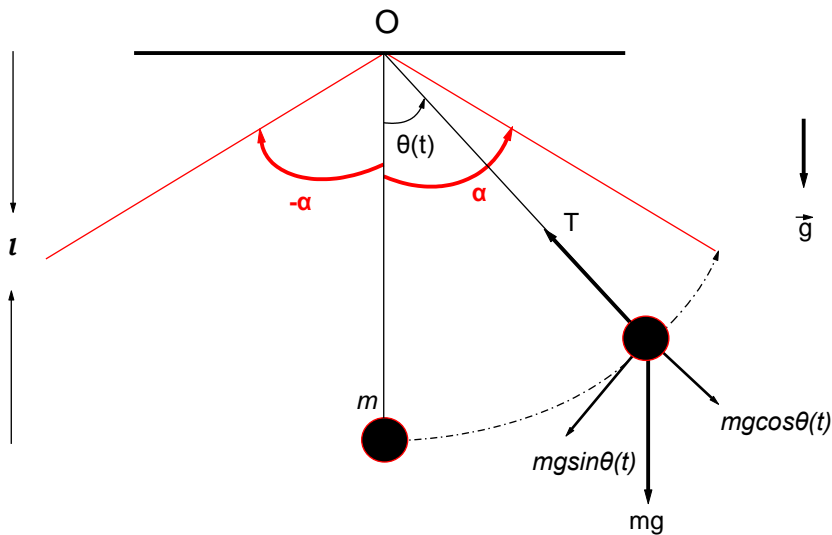

FIGURE 1. The physical pendulum. The vertical position of the chord corresponds to zero potential energy, while α stands for the angular amplitude of the swing.

The motion corresponding to (3) is isochronic, meaning that the period is independent of the amplitude and constant for a given pendulum. The reason for this comes from the Langrangian of the pendulum, which is then also linearized in the small angle regime, ($\sin\theta \cong \theta$), as $L(\theta,\dot{\theta}) = T(\dot{\theta}) - U(\theta) = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\theta^2$. Thus, if we carry out a transformation in which the angular displacement is changed by a factor λ while the time interval remains unchanged, then the Lagrangian is simply multiplied by the constant λ^2 and the equations of motion are unaltered. Thus for a given linear pendulum the equations of motion permit a series of geometrically similar paths where the times of motion between corresponding points are in unit ratio. This similarity vanishes for large amplitudes and this is clearly reflected on the dependence of the period on the amplitude and subsequently on the energy as well.

Since the elliptic integral is not easily calculated, many approximation schemes have been developed over the years for the calculation of the period of swing: see [12] and references therein. In this way many of these approximations lead to interesting applications, as is for example the use of the physical pendulum as part of the set up for measuring acoustic impedance, via the use of the Struve function [13].

THE QUANTUM PENDULUM

In order to construct the time independent Schrödinger equation for the pendulum we start from the Lagrangian $L(\theta,\dot{\theta}) = T(\dot{\theta}) - U(\theta) = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$ and find the system's generalized momentum [10] as

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \Rightarrow p_\theta = ml^2\dot{\theta} \tag{6}$$

Thus the Hamiltonian can be expressed as a function of the generalized variables θ, p_θ

$$H(\theta,\dot{\theta}) = T(\dot{\theta}) + U(\theta) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) \Rightarrow$$

$$H_g(\theta,p_\theta) = \frac{1}{2}ml^2\left(\frac{p_\theta}{ml^2}\right)^2 + mgl(1-\cos\theta) \Rightarrow \tag{7}$$

$$H_g(\theta,p_\theta) = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)$$

Then the operator $\hat{H}$ corresponding to the classical Hamiltonian is found by applying the convention rule and replacing the canonical momentum p_θ by $-i\hbar\frac{\partial}{\partial\theta}$, to take

$$\hat{H} = -\frac{\hbar^2}{2ml^2}\frac{\partial^2}{\partial\theta^2} + mgl(1-\cos\theta) \Rightarrow$$

$$\hat{H} = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2} + I\omega^2(1-\cos\theta) \tag{8}$$

where we have substituted $I = ml^2$ for the moment of inertia and $\omega^2 = \dfrac{g}{l}$ for the square of the classical frequency of swinging. Thus the Schrödinger equation reads

$$-\frac{\hbar^2}{2I}\frac{\partial^2 \Psi(\theta)}{\partial \theta^2} + I\omega^2(1-\cos\theta)\Psi(\theta) = E\Psi(\theta) \tag{9}$$

The wavefunction $\Psi(\theta)$ has to be single-valued, meaning periodic in θ of period 2π: $\Psi(\theta + 2\pi) = \Psi(\theta)$. In addition we can write (9) as

$$\frac{\partial^2 \Psi(\theta)}{\partial \theta^2} + \left(\frac{2I}{\hbar^2}\left(E - I\omega^2\right) + 2\left(\frac{I\omega}{\hbar}\right)^2 \cos\theta\right)\Psi(\theta) = 0 \tag{10}$$

and by further substituting $\theta = 2\varphi$ as

$$\frac{1}{4}\frac{\partial^2 \Psi(\phi)}{\partial \phi^2} + \left(\frac{2I}{\hbar^2}\left(E - I\omega^2\right) + 2\left(\frac{I\omega}{\hbar}\right)^2 \cos 2\phi\right)\Psi(\phi) = 0 \Rightarrow$$

$$\frac{\partial^2 \Psi(\phi)}{\partial \phi^2} + \left(\frac{8I}{\hbar^2}\left(E - I\omega^2\right) + 8\left(\frac{I\omega}{\hbar}\right)^2 \cos 2\phi\right)\Psi(\phi) = 0 \tag{11}$$

where for simplicity we have used the same symbol for the wave-function. The last form is that of a Mathieu equation [14] whose standard form is

$$\frac{d^2 y}{dx^2} + \left(a - 2q\cos 2x\right)y = 0 \tag{12}$$

with

$$a = \frac{8I}{\hbar^2}\left(E - I\omega^2\right) > 0 \text{ and } q = -4\left(\frac{I\omega}{\hbar}\right)^2 < 0 \tag{13}$$

The periodicity of the wave-function now demands $\Psi(2\phi + 2\pi) = \Psi(2\phi) \Rightarrow \Psi\left(2(\phi+\pi)\right) = \Psi(2\phi)$ which means that the wave-function is periodic of period π as a function of φ. Such solutions are the Mathieu functions of even order [14], namely the $ce_{2m}(\phi)$ and $se_{2m+2}(\phi)$ for m=0,1,2,.. However, as was pointed out in [15], equation (11) corresponds to a Mathieu equation with negative q, and not positive as it should, and so we have to further perform a last change of variables of the form $\phi \to \dfrac{\pi}{2} - \theta$. The normalized solutions of the Schrödinger equation (11) are given by the following relations [15]:

$$\Psi_o^{(e)}(\theta, -q) = \frac{1}{\sqrt{2\pi}} ce_o\left(\frac{\pi - \theta}{2}, q\right)$$

$$\Psi_{2m}^{(e)}(\theta, -q) = \frac{1}{\sqrt{\pi}}(-1)^m ce_{2m}\left(\frac{\pi - \theta}{2}, q\right) \tag{14}$$

$$\Psi_{2m}^{(o)}(\theta, -q) = \frac{1}{\sqrt{\pi}}(-1)^{m+1} se_{2m}\left(\frac{\pi - \theta}{2}, q\right)$$

where the superscripts *e* and *o* denote even and odd solutions in *θ* respectively. By assuming trigonometric expansions for the eigenfunctions we can compute both the eigenvalues (known as characteristic values) and the related eigenfunctions. However the calculations are rather tedious since they involve numerical calculations based on certain three-term recursion relations or appropriate definitions of quantities as continued fractions [15]. An initial guess of the wave function is also needed.

Things become extremely simple however, if we assume a parabolic potential ($\sin\theta \cong \theta$) to write (9) in the form

$$-\frac{\hbar^2}{2I}\frac{\partial^2\Psi(\theta)}{\partial\theta^2} + 2I\omega^2\sin^2(\theta/2)\Psi(\theta) = E\Psi(\theta)$$

<div align="center">approximated as</div>   (15)

$$-\frac{\hbar^2}{2I}\frac{\partial^2\Psi(\theta)}{\partial\theta^2} + \frac{1}{2}I\omega^2\theta^2\Psi(\theta) = E\Psi(\theta)$$

The above correspond to the harmonic oscillator problem whose eigenfunctions are given in terms of the Hermite polynomials [16] while its eigenvalues take the simple form

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$ (16)

Since only the $\Delta n = \pm 1$ transitions are allowed [16], *ω* is equal to the corresponding photon frequency and is common for all the possible values of energy. This is the quantum equivalent to classical isochronism that was mentioned in the previous section. In this way we can calculate the classical frequency of swinging through the difference of two successive quantum energy levels.

THE SEMICLASSICAL PATH INTEGRAL APPROACH

The path integral formulation can be applied to the construction of a system's Green's function, the later being the Fourier transformed in time of the propagator [17]. In its semiclassical version which is based on a periodic orbit theory [18,19], the Green's function for one dimensional propagation is accomplished by taking account of all possible changes in phase of the wave-function. However this is not trivial at all since it encloses the contribution of all possible changes of phase when a turning point is met either in the allowed or in the forbidden regions of motion, for an infinite set of possible paths.

The above mentioned pioneering work inspired many others to further improve the method and analytically solve interesting one dimensional problems. Among those, Holstein and Swift [20] and Holstein alone [21] showed how *Gsc(E)* , which is the semiclassical fixed energy transmission amplitude, can be used to achieve analytic continuation of the propagator to forbidden regions, and furthermore established its connection to propagation and to reflection. For example when *Gsc(E)* is calculated for propagation between two points in a classically allowed region of motion, the perturbed complex energy spectrum of the later is revealed [22]. The central result for the calculation of the transmission amplitude via an infinite set of paths that the particle follows, can be written in compact form as

$$Gsc(E) \;=\; \frac{\mathrm{m}}{2\pi \overline{k(r_1,r_2)}} \sum_{j=1}^{\infty} \left\{ \prod_{i=1}^{N(j)} a_{ij} \right\} \tag{17}$$

In the above equation $\overline{k(r_1,r_2)}$ is a non local wave number of the particle connecting the initial and the final point of propagation and defined by $\overline{k(r_1,r_2)} = \sqrt{k(r_1)k(r_2)}$ where $k(r) = \hbar^{-1}\sqrt{2m(E - V(r))}$, with E standing for the energy and $V(r)$ for the potential function. The index j corresponds to a particular path, while the index i corresponds to a certain event along the path. Therefore, the symbol a_{ij} represents each i event factor that contributes to the jth path. Their total number is $N(j)$ and depends on the path. These event factors are of two types. One type represents propagation and the other represents reflection. The a_{ij} propagation event factors describe propagation (from α to β) in an either allowed (given by $\exp\left[i\int_{\alpha}^{\beta} k(r)dr\right] \equiv e^{ik}$, where $k \neq k(r)$), or in a forbidden region of motion (given by $\exp\left[-\left|\int_{\alpha}^{\beta} \kappa(r)dr\right|\right] \equiv e^{-\kappa}$ with $\kappa(r) = \hbar^{-1}\sqrt{2m(V(r) - E)}$, where $\kappa \neq \kappa(r)$), while the a_{ij} reflection event factors describe reflections from turning points, (-i for reflection from a turning point in an allowed region, + $i/2$ for reflection in a forbidden region , and –1 for reflection from an infinite barrier). Both k and κ are dimensionless quantities and should not be confused with the corresponding position dependent wave numbers, although we have kept the same symbol in order to emphasize their origin. Thus the product $\prod_{i=1}^{N(j)} a_{ij}$ gives the unique amplitude for each possible path for going from r_1 to r_2 in constant energy. The sum in (17) extends to the infinity of possible paths connecting the space points r_1 and r_2. The above mentioned rules are depicted in the figure that follows.
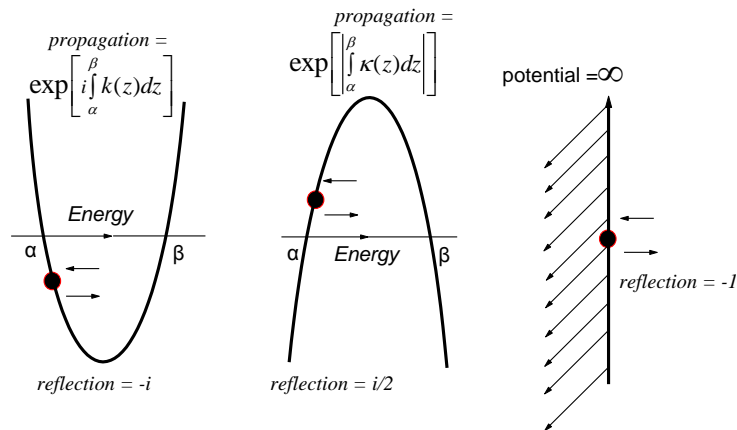

FIGURE 2. The rules for the construction of the path integral amplitudes through the α$_{ij}$ event factors. The first motion is in a classically allowed region, the second in a classically forbidden region and the third case describes reflection from an infinite barrier. The direction before the reflection is always from right to left and is reversed after meeting the turning point.

The above described method can also be found in standard textbooks of path integrals, or quantum tunneling as well, [23,24].

It is readily seen that the calculation of the overall transmission amplitude depends on the topology of the potential function and on the nature of the turning points. Since there is an infinity of paths traversing both the allowed and the forbidden regions, it is very crucial to include all of them in the calculation by performing correct the rather complicated combinatorics. Then, and since each path repeats itself, the infinite class of paths can be summed to constitute geometric progressions, from which the analytic properties of *Gsc(E)* can be recognized directly. For the present requirement of computing the overall transmission amplitude, the points r_1 and r_2 are in the classically allowed region of motion of the pendulum, which is the area between angles $-\alpha$ and α shown in Figure 1.

THE CONSTRUCTION OF THE PENDULUM'S GREEN'S FUNCTION

Since the pendulum has only one degree of freedom it can be treated as an one dimensional physical system. Thus we can apply the formulation of the semiclassical Green's function that was described in the previous chapter, with only a few changes in notation. In particular the mass factor comes as the moment of inertia instead of the mass alone and we use the angle variable θ instead of the space variable *r*. In addition we should sketch the one dimensional potential as in Figure 3 that follows, in order to properly describe the hindered rotation of the pendulum:
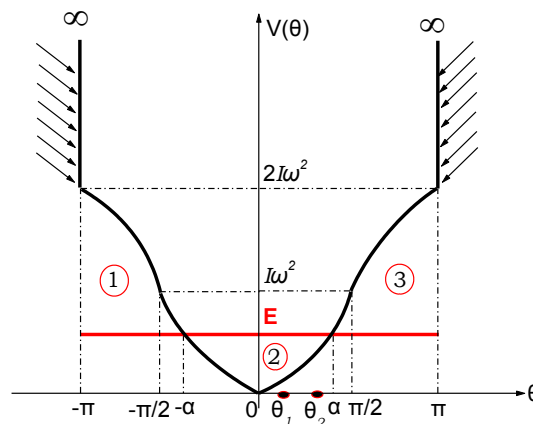

FIGURE 3. The potential of the quantum pendulum for hindered rotation. In order to exclude free rotation an infinite barrier is supposed at angle π (-π). There are three regions of motion: the classically accessible region 2, and the classically inaccessible regions 1 and 3, as these are structured by the specific value of the energy E and are separated from each other by the turning points α and –α.

Figure 3 describes the induced topology of the potential as this is determined by the specific value of the energy. There are two turning points, meaning angles $-\alpha$ and α, while an infinite rotational barrier is supposed at angle π ($-\pi$), in order to forbid free rotation. For example we can imagine an elastic wall at this position. In this way we have three regions of motion, one classically allowed (region 2) and two classically forbidden (regions 1 and 3). In order to construct the overall transition amplitude for propagation between points θ_1 and θ_2 of region 2, we separate the problem in smaller ones. For this we write $Gsc(E)$ as a sum of transition amplitudes involving specific regions of motion each time, of the form:

$$G_{sc}(E) = G^2 + G^{2,1} + G^{2,3} + G^{2,1,3} \tag{18}$$

where $G^{\rho,\sigma}$ for example denotes the amplitude for propagation involving regions ρ and σ only, in all possible ways. In the paragraphs that follow we separately develop each amplitude providing the basic steps.

i)   Amplitude G^2

It involves propagation inside region 2. It is constructed by fundamental amplitudes $\tilde{A}_{x,y}$ that connect points x and y in a straight path, and by amplitudes $A_{x,y}$, (we use the A letter for the classically accessible region of motion 2), that connect x and y with multiple repetitions (including reflections) in all possible ways, following the rules of the event factors that were given in the previous section. In this way we can write

$$G^2 = \tilde{A}_{\theta_1,\theta_2} + \tilde{A}_{\theta_1,\alpha}\left\{A_{a,\alpha}\tilde{A}_{a,\theta_2} + A_{a,-\alpha}\tilde{A}_{-a,\theta_2}\right\} + \tilde{A}_{\theta_1,-\alpha}\left\{A_{-a,\alpha}\tilde{A}_{a,\theta_2} + A_{-a,-\alpha}\tilde{A}_{-a,\theta_2}\right\} \Rightarrow$$

$$G^2 = \tilde{A}_{\theta_1,\theta_2} + \tilde{A}_{\theta_1,\alpha}A_{a,\alpha}\left\{\tilde{A}_{a,\theta_2} - ie^{i\mu}\tilde{A}_{-a,\theta_2}\right\} + \tilde{A}_{\theta_1,-\alpha}A_{-a,-\alpha}\left\{\tilde{A}_{-a,\theta_2} - ie^{i\mu}\tilde{A}_{a,\theta_2}\right\} \Rightarrow \tag{19}$$

$$G^2 = \tilde{A}_{\theta_1,\theta_2} + A_{a,\alpha}\left\{\tilde{A}_{\theta_1,\alpha}\left(\tilde{A}_{a,\theta_2} - ie^{i\mu}\tilde{A}_{-a,\theta_2}\right) + \tilde{A}_{\theta_1,-\alpha}\left(\tilde{A}_{-a,\theta_2} - ie^{i\mu}\tilde{A}_{a,\theta_2}\right)\right\}$$

where we define the following quantities

$$\tilde{A}_{x,y} = \exp\left[i\frac{\sqrt{2\mathrm{I}}}{\hbar}\int_x^y \sqrt{E-V(\theta)}d\theta\right]$$

$$\tilde{A}_{-a,a} = \exp\left[i\frac{\sqrt{2\mathrm{I}}}{\hbar}\int_{-a}^a \sqrt{E-V(\theta)}d\theta\right] \equiv e^{i\mu}$$

$$A_{-a,-a} = A_{a,a} = -i + (-i)e^{i\mu}(-i)e^{i\mu}(-i) + (-i)e^{i\mu}(-i)e^{i\mu}(-i)e^{i\mu}(-i)e^{i\mu}(-i) + ... \Rightarrow \tag{20}$$

$$A_{-a,-a} = A_{a,a} = -i\left[1 + (-ie^{i\mu})^2 + (-ie^{i\mu})^4 + ...\right] \Rightarrow$$

$$A_{-a,-a} = A_{a,a} = -\frac{i}{1+e^{2i\mu}}$$

Thus the G^2 amplitude takes the following form

$$G^2 = \tilde{A}_{\theta_1,\theta_2} - \frac{i}{1+e^{2i\mu}}\left\{\tilde{A}_{\theta_1,\alpha}\left(\tilde{A}_{a,\theta_2} - ie^{i\mu}\tilde{A}_{-a,\theta_2}\right) + \tilde{A}_{\theta_1,-\alpha}\left(\tilde{A}_{-a,\theta_2} - ie^{i\mu}\tilde{A}_{a,\theta_2}\right)\right\} \tag{21}$$

ii)   Amplitudes $G^{2,1}$, $G^{2,3}$

It is quite obvious that the above amplitudes are equal to each other due to symmetry reasons. We use the letter B for the symbolism of the amplitudes in the forbidden region 1. In addition the star as a superscript denotes the fact that the amplitude includes at least one propagation event and not just a reflection factor. Since $-\alpha$ is the border between the two regions of motion the goal is to reach angle $-\alpha$ in all possible ways while staying in region 2, and then interchange the two regions in every possible way. According to these we can write

$$G^{2,1} = A_{\theta_1,-\alpha}\left\{ B^*_{-a,-\alpha}A_{-a,-\alpha} + \left(B^*_{-a,-\alpha}A_{-a,-\alpha}\right)^2 + ...\right\}\left(\tilde{A}_{-a,\theta_2} + \tilde{A}_{-\alpha,\alpha}\tilde{A}_{a,\theta_2}\right) \tag{22}$$

which transforms to the following due to the geometric progression that appears

$$G^{2,1} = A_{\theta_1,-\alpha}\left\{ \frac{B^*_{-a,-\alpha}A^*_{-a,-\alpha}}{1-B^*_{-a,-\alpha}A^*_{-a,-\alpha}}\right\}\left(\tilde{A}_{-a,\theta_2} + \tilde{A}_{-\alpha,\alpha}\tilde{A}_{a,\theta_2}\right) \tag{23}$$

and where we define the following quantities

$$A^*_{-a,-a} = e^{i\mu}(-i)e^{i\mu} + e^{i\mu}(-i)e^{i\mu}(-i)e^{i\mu}(-i)e^{i\mu} + ... \Rightarrow$$

$$A^*_{-a,-a} = -\frac{ie^{2i\mu}}{1+e^{2i\mu}}$$

$$\tilde{B}_{-\pi,-a} = \exp\left[-\frac{\sqrt{2I}}{\hbar}\int_{-\pi}^{-a}\sqrt{V(\theta)-E}\,d\theta\right] \equiv e^{-\kappa} \tag{24}$$

$$B^*_{-a,-a} = e^{-\kappa}(-1)e^{-\kappa} + e^{-\kappa}(-1)e^{-\kappa}(i/2)e^{-\kappa}(-1)e^{-\kappa} + ... \Rightarrow$$

$$B^*_{-a,-a} = -\frac{2e^{-2\kappa}}{2+ie^{-2\kappa}}$$

Putting all these in equation (22) and performing the algebra we get the following result for the amplitude

$$G^{2,1} = \left\{ \frac{2e^{4i\mu}e^{-2\kappa}\left(\tilde{A}_{-a,\theta_2} + \tilde{A}_{-\alpha,\alpha}\tilde{A}_{a,\theta_2}\right)\left(-ie^{i\mu}\tilde{A}_{\theta_1,\alpha} + \tilde{A}_{\theta_1,-\alpha}\right)}{\left(1+e^{2i\mu}\right)\left[\left(1+e^{2i\mu}\right)\left(2+ie^{-2\kappa}\right) - 2ie^{-2\kappa}e^{2i\mu}\right]}\right\} \tag{25}$$

The total amplitude for propagation in the classically allowed and one of the two classically forbidden regions of motion will be twice of the above for the reasons previously explained, and so we finally get

$$G^{2,(1or3)} = G^{2,1} + G^{2,3} = \left\{ \frac{4e^{4i\mu}e^{-2\kappa}\left(\tilde{A}_{-a,\theta_2} + \tilde{A}_{-\alpha,\alpha}\tilde{A}_{a,\theta_2}\right)\left(-ie^{i\mu}\tilde{A}_{\theta_1,\alpha} + \tilde{A}_{\theta_1,-\alpha}\right)}{\left(1+e^{2i\mu}\right)\left[\left(1+e^{2i\mu}\right)\left(2+ie^{-2\kappa}\right) - 2ie^{-2\kappa}e^{2i\mu}\right]}\right\} \tag{26}$$

    iii)   Amplitude $G^{2,1,3}$

It order to properly construct the amplitude for paths involving all regions of motion we think as follows: first of all we should reach angle $-\alpha$ in every possible way while constantly staying at region 2, which constitutes amplitude $A_{\theta_1,-\alpha}$, and then interchange regions 1 and 2 in every possible way starting and ending up to angle $-\alpha$, this is the amplitude $X^{-\alpha}(1,2)$ to be defined. We reach angle α in a straight path and then interchange regions 3 and 2 in every possible way starting and ending up to angle α, this is amplitude $X^{\alpha}(3,2)$, and we infinitely repeat this process. We interchange for the last time regions 1 and 2, and finally reach angle θ_2 while remaining in region 2. Finally we should multiply the above overall amplitude with a factor of two, since we could equally well start with the interchange of regions 2 and 3, and proceed analogously. The above are mathematically described by the following equation

$$G^{2,1,3} = 2A_{\theta_1,-\alpha} \begin{Bmatrix} X^{-\alpha}(1,2)(-i)\tilde{A}_{-a,\alpha}X^{\alpha}(3,2)(-i)\tilde{A}_{a,-\alpha} + \\ \left[X^{-\alpha}(1,2)(-i)\tilde{A}_{-a,\alpha}X^{\alpha}(3,2)(-i)\tilde{A}_{a,-\alpha} \right]^2 +.. \end{Bmatrix} X^{-\alpha}(1,2)(-i)A_{-a,\theta_2} \Rightarrow$$

$$G^{2,1,3} = 2iA_{-a,\theta_2}A_{\theta_1,-\alpha} \frac{\left(X^{-\alpha}(1,2)\tilde{A}_{-a,\alpha} \right)^2}{1+\left(X^{-\alpha}(1,2)\tilde{A}_{-a,\alpha} \right)^2} X^{-\alpha}(1,2) \Rightarrow \qquad (27)$$

$$G^{2,1,3} = 2iA_{-a,\theta_2}A_{\theta_1,-\alpha} \frac{X^{-\alpha}(1,2)}{1+\left(X^{-\alpha}(1,2)\tilde{A}_{-a,\alpha} \right)^{-2}}$$

where we have used the fact that the two "doublet" amplitudes $X^{-\alpha}(1,2)$ and $X^{\alpha}(3,2)$ are equal to each other due to symmetry reasons. In fact each doublet is given by the following formula

$$X^{-\alpha}(1,2) = X^{\alpha}(3,2) = B^{*}_{-a,-\alpha}A_{-a,-\alpha} + \left(B^{*}_{-a,-\alpha}A_{-a,-\alpha} \right)^2 +... \Rightarrow$$

$$X^{-\alpha}(1,2) = X^{\alpha}(3,2) = \frac{B^{*}_{-a,-\alpha}A^{*}_{-a,-\alpha}}{1-B^{*}_{-a,-\alpha}A^{*}_{-a,-\alpha}} \qquad (28)$$

Putting equations (24) and (28) in (27) and performing the tedious algebra we get the following result for the amplitude $G^{2,1,3}$

$$G^{2,1,3} = \frac{2ie^{-6\kappa}e^{12i\mu}\left(\tilde{A}_{-a,\theta_2} - ie^{i\mu}\tilde{A}_{a,\theta_2} \right)\left(\tilde{A}_{\theta_1,-a} - ie^{i\mu}\tilde{A}_{\theta_1,\alpha} \right)}{\left(1+e^{2i\mu}\right)^2 \left[(1+e^{2i\mu})(2+ie^{-2\kappa}) - 2ie^{-2\kappa}e^{2i\mu} \right]\left\{ \left[(1+e^{2i\mu})(2+ie^{-2\kappa}) - 2ie^{-2\kappa}e^{2i\mu} \right]^2 - 4e^{-4\kappa}e^{6i\mu} \right\}}$$

$$(29)$$

CALCULATION OF THE QUANTUM CORRECTED CLASSICAL SWING PERIOD

The total propagation amplitude as this is calculated by the sum of equations (21), (26) and (29), is constructed by a regular (non pole) and three pole (fractional) terms. It is clear that the pole terms contain all the bits of interesting information for the problem under study, while the regular term provides the general background of propagation. In fact the major contribution to propagation is achieved when the denominator of a pole term approaches (if not equal) to zero. In this way the energy spectrum of the system is revealed, since it corresponds to the energy poles of the propagation amplitude, and the Green's function itself can be written as a sum over the resonance energy states that the pendulum potential can support, [19].

Amplitude G^2 involves propagation inside classically accessible region 2 only, and so its energy spectrum resembles the one of a bound problem, like the harmonic oscillator (H.O.S.). For narrow states, which are states close to the bottom of the well, we get a closer approximation to the H.O.S. In order to calculate the spectrum we seek for the zeros of $1+e^{2i\mu}$, which is the denominator of the pole part of the amplitude G^2. Thus, in this framework, energy poles E_n will appear whenever

$$1+e^{2i\mu}=0 \Rightarrow \mu = n\pi + \pi/2 \qquad (30)$$

Let us calculate μ for the harmonic potential of the form $V(\theta)=\frac{1}{2}I\omega^2\theta^2$. According to equation

(20) we have $\mu(E)=\frac{\sqrt{2I}}{\hbar}\int_{-\alpha}^{\alpha}\sqrt{E-\frac{1}{2}I\omega^2\theta^2}d\theta$. The integral is elementary and we get $\mu(E)=\frac{\pi E}{\hbar\omega}$.

Imposing the pole condition of equation (30) we get $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$, and so we reveal the H.O.S.

The classical swing period can be extracted from the spectrum in two ways. First, it is given in terms of the derivative of $\mu(E)$ with respect to energy as

$$T=2\hbar\frac{\partial\mu(E)}{\partial E}\equiv 2\breve{\tau}(E) \qquad (31)$$

where we have defined the time scale $\breve{\tau}(E)$ for the classically accessible region of motion. The reason for eq. (31) to happen comes from the fact that according to the physics of path integration $\mu(E)$ is the action phase factor for propagation in the classically accessible region of motion with constant energy E, and so is equal to $E\cdot\tau/\hbar$, where τ is the time of propagation, [21]. Since τ is half the classical period, equation (31) follows. Second and since the energy spectrum depends only on quantum number n, it can be given in terms of the energy derivative with respect to quantum number n,

$$T=\frac{2\pi\hbar}{\partial E_n/\partial n} \qquad (32)$$

where equation (32) comes from the expansion of the energy eigenvalues about the central value (when a wave packet is considered), in terms of which we can take the classical period, as well as the revival, and superrevival times of the wave packet, [25]. Equations (31) and (32) become

equivalent to each other under the action of condition (30), since then μ becomes a function of the quantum number n, and so

$$(\breve{\tau}(\mathrm{E}) \equiv) \hbar \frac{\partial \mu(E)}{\partial E} = \hbar \frac{\partial \mu(E)}{\partial n} \frac{\partial n}{\partial E} = \pi \hbar \frac{\partial n}{\partial E} \tag{33}$$

For the case under study equation (32) is simply transformed to

$$T(E_n) = \frac{2\pi\hbar}{E_{n+1} - E_n} \tag{34}$$

As far as the harmonic potential is concerned, equations (31) and (32) give the classical result, meaning equation (3) for the classical swing period, where the quantum effects are absent. For the case of the real trigonometric potential, equation (31) can be written as

$$T = 2\hbar \frac{\partial \mu(E)}{\partial E} = 2\hbar \left[\frac{\sqrt{2I}}{\hbar} \frac{1}{2\sqrt{mgl}} \int_{-a}^{a} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} \right] = \left[2\sqrt{\frac{l}{2g}} \int_{-a}^{a} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} \right] \tag{35}$$

which of course coincides with the classical result of eq. (4) and (5), and gives the swing period in terms of the elliptic integral. Again the quantum effects are absent and this is due to the fact that the amplitude G^2 is constructed by taking in account only the classically accessible region of motion. However the spectrum is no more continuous since condition (30) quantizies the spectrum, in a way equivalent to the W.K.B. approximation, [16].

In order to calculate the classical swing period as this is perturbed from its classical value due to the quantum tunneling effect, we should look at the pole terms of the amplitudes $G^{2,(1 or 3)}$ and $G^{2,1,3}$ that emerge through propagation in the classically inaccessible regions 1 and 3. Both of these maximally increase their magnitude when their denominator approaches zero. Besides equation (30) this happens when the following condition is fulfilled

$$(1 + e^{2i\mu})(2 + ie^{-2\kappa}) - 2ie^{-2\kappa}e^{2i\mu} = 0 \tag{36}$$

However, equation (36) has not real solutions, meaning that there do not exist real values of the energy to satisfy it. Condition (30) makes (36) almost zero, besides the remaining term $2ie^{-2\kappa}$, which is exponentially small in magnitude. If we expand equation (36) in first order about the energy poles E_n of the classical accessible region 2, we get

$$Z_n \cong E_n + \frac{\hbar e^{-2\kappa}\left(\breve{\tau}(E_n) + e^{-2\kappa}\hat{\tau}(E_n)\right)/2}{\breve{\tau}^2(E_n)e^{-4\kappa} + \left(\breve{\tau}(E_n) + e^{-2\kappa}\hat{\tau}(E_n)\right)^2} + \frac{i}{2} \frac{\hbar e^{-4\kappa}\breve{\tau}(E_n)}{\breve{\tau}^2(E_n)e^{-4\kappa} + \left(\breve{\tau}(E_n) + e^{-2\kappa}\hat{\tau}(E_n)\right)^2} \tag{37}$$

where we have defined time scale $\hat{\tau}(\mathrm{E})$ for the classically inaccessible region of motion in an analogous to equation (31) way, meaning

$$\hat{\tau}(\mathrm{E}) \equiv \hbar \frac{\partial \kappa(E)}{\partial E} \tag{38}$$

The perturbed energy poles Z_n have been calculated by taking in account the tunneling effect in the classically forbidden regions 1 and 3. In this way and according to our previous analysis, the difference between the real part of two successive perturbed energy poles, will provide the quantum corrected swing frequency. Writing the real part of the perturbed energy poles as a function of the unperturbed energy poles, meaning

$$\mathrm{Re}\, Z_n = E_n + f\left(E_n\right) \tag{39}$$

and introducing the perturbed swing frequency Ω as

$$\operatorname{Re} Z_{n+1} - \operatorname{Re} Z_n \equiv \hbar\Omega \tag{40}$$

we get the following equation

$$\operatorname{Re} Z_{n+1} - \operatorname{Re} Z_n = E_{n+1} - E_n + \left(f(E_{n+1}) - f(E_n) \right) \Rightarrow \tag{41}$$

$$\hbar\Omega = \hbar\omega(E_n) \left(1 + \frac{\partial f(E_n)}{\partial E_n} \right)$$

By keeping only the dominant terms in equation (37), equation (41) gives the following result for the quantum shift of the frequency

$$\Delta\omega = -\frac{\omega(E_n)}{T(E_n)} 2e^{-2\kappa} \left(\hat{\tau}(E_n) + \hbar \frac{\partial \left(\ln T(E_n) \right)}{\partial E_n} \right) \tag{42}$$

Thus, the quantum corrected period of swing can be easily calculated. Using the index *quan* for quantum and *cl* for classical in order to emphasize our result, we can write the following formula

$$T_{quan}(E_n) = T_{cl}(E_n) \exp\left\{ \frac{2e^{-2\kappa(E_n)}}{T_{cl}(E_n)} \left(2\hat{\tau}(E_n) + \hbar \frac{\partial \left(\ln T_{cl}(E_n) \right)}{\partial E_n} \right) \right\} \tag{43}$$

which is the central result of this work. Due to the presence of both the exponentially small term $e^{-2\kappa(E_n)}$ and Planck's constant, quantum and classical period are nearly equal in magnitude. Quantum corrections, as these are contained in the above relation, come from three different sources. First the existence of the quantum tunneling effect through the exponential $e^{-2\kappa(E_n)}$ term, second the dependence of the tunneling effect on energy through the induced time scale $\hat{\tau}(E_n)$, and third the dependence of the classical period on energy (present in the classical analysis as well), through its logarithmic derivative $\dfrac{\partial \left(\ln T_{cl}(E_n) \right)}{\partial E_n}$.

Let us look in some detail the physical meaning of the tunneling time scale $\hat{\tau}(E_n)$. The energy dependence of the classical action inside the potential barrier, meaning quantity $\hbar\kappa(E) = \sqrt{2I} \int\limits_{-\pi}^{-\alpha(E)} d\theta \sqrt{V(\theta) - E}$, comes not only from the integrand but from the upper limit of integration as well. The action phase factor may be written as

$$\kappa(E) = \frac{\sqrt{2I}}{\hbar} \int\limits_{-\pi}^{-\alpha(E)} T(\theta, E) d\theta = $$

$$\frac{\sqrt{2I}}{\hbar} \left\{ \Phi[-\alpha(E), E] - \Phi[-\pi, E] \right\} \tag{44}$$

where of course

$$\frac{\partial \Phi(\theta, E)}{\partial \theta} = T(\theta, E) \tag{45}$$

However, and since the upper limit of integration is the classical turning point, the following result holds,

$$T[-\alpha(E), E] = 0 \tag{46}$$

which leads to the

$$\frac{d\kappa(E)}{dE} = -\sqrt{\frac{I}{2\hbar^2}} \int_{-\pi}^{-\alpha(E)} d\theta \frac{1}{\sqrt{V(\theta)-E}} \tag{47}$$

which can be further transformed to the following

$$\frac{d\kappa(E)}{dE} = \sqrt{\frac{I}{2\hbar^2}} \left\{ \int_{E}^{2I\omega^2} \frac{d\theta}{dV} \frac{1}{\sqrt{V-E}} dV \right\} \tag{48}$$

In order to eliminate E from the right term we introduce the weight function $(E-\lambda)^{-1/2}$ and perform integration over E in both terms. It is clear that the parameter λ has energy dimensions. Equation (48) is then transformed to:

$$\int_{\lambda}^{2I\omega^2} dE \frac{d\kappa(E)}{dE} \frac{1}{\sqrt{E-\lambda}} = \sqrt{\frac{I}{2\hbar^2}} \int_{\lambda}^{2I\omega^2} dE \int_{E}^{2I\omega^2} \left(\frac{d\theta}{dV}\right) \frac{1}{\sqrt{V-E}} \frac{1}{\sqrt{E-\lambda}} dV \tag{49}$$

The double integral can be easily handled via a change in the order of integration to take the

$$\int_{\lambda}^{2I\omega^2} dE \frac{d\kappa(E)}{dE} \frac{1}{\sqrt{E-\lambda}} = \sqrt{\frac{\mu}{2\hbar^2}} \int_{\lambda}^{2I\omega^2} dV \int_{\lambda}^{V} \left(\frac{d\theta}{dV}\right) \frac{1}{\sqrt{V-E}} \frac{1}{\sqrt{E-\lambda}} dE \tag{50}$$

The integral over E is then elementary and equal to π, and so we take:

$$\int_{\lambda}^{2I\omega^2} dE \frac{d\kappa(E)}{dE} \frac{1}{\sqrt{E-\lambda}} = \pi \sqrt{\frac{I}{2\hbar^2}} \int_{\lambda}^{2I\omega^2} dV \left(\frac{d\theta}{dV}\right) \tag{51}$$

while the last integral is trivial and gives

$$\int_{\lambda}^{2I\omega^2} dE \frac{d\kappa(E)}{dE} \frac{1}{\sqrt{E-\lambda}} = \pi \sqrt{\frac{I}{2\hbar^2}} \left\{ \pi - \theta(\lambda) \right\} \Rightarrow$$

$$\int_{\lambda}^{2I\omega^2} \hat{\tau}(E) \frac{dE}{\sqrt{E-\lambda}} = \pi \sqrt{\frac{I}{2}} \left\{ \pi - \theta(\lambda) \right\} \tag{52}$$

Since $\hat{\tau}(E)$ is defined only inside the classically inaccessible region (1 or 3), it is clear that $\lambda \geq V(\alpha)$. Equation (52) then, describes the way that the knowledge of the function $\hat{\tau}(E)$ permits the knowledge of the angular position inside the potential barrier as a function of the potential. In addition $\hat{\tau}(E)$ appears in both the real and imaginary energy shift of the perturbed energy poles in equation (37). In a previous work of us [26], the later was connected with what was defined as energy indeterminacies during the tunneling process.

CONCLUDING REMARKS

In the present work we derived an improved and prototype formula for the swing period of the classical pendulum, by taking in account the pure quantum mechanical tunneling effect via semi-classical path integration. Our central result is contained in equation (43) where the tunneling effect is present through the action phase factor $\kappa(E)$ and its derivative with respect to energy. Both the $\mu(E)$ and $\kappa(E)$ action phase factors in eq. (43), defined in the classically accessible and classically forbidden regions of motion respectively, introduce a different time scale of the physical system under study. The first order energy derivative of $\mu(E)$ reproduces the classical formula for the swing period in terms of the elliptic integral, while its second order derivative reflects the quantum equivalent to classical isochronism. On the other hand the energy derivative of the $\kappa(E)$ action phase factor introduces the time scale $\hat{\tau}(E_n)$ whose knowledge is equivalent to the knowledge of the position of the particle during tunneling, as a function of the potential.

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